Eur. Phys. J. D **33**, 285–296 (2005)
DOI: 10.1140/epjd/e2005-00032-x

THE EUROPEAN
PHYSICAL JOURNAL D

Geometric phase and entanglement for massive spin-1 particles

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Received 15 October 2004 / Received in final form 28 December 2004
Published online 5 April 2005 –

Abstract. The cyclic evolution of a spin-1 system is studied under the spin-spin interaction between the transverse and the longitudinal states. The eigenstates of the systems are obtained by generalized and extended Jordan-Wigner transformation with an angle ϕ described the path of particle propagation. According to the wave functions of time evaluation for many-particle systems, the entanglement effects and geometric phase are observed. The systems with more than two particles, in contrast to the two particle system, evolve in time with two parameters.

PACS. 03.65.Bz Foundations, theory of measurement, miscellaneous theories (including Aharonov-Bohm effect, Bell inequalities, Berry's phase) – 03.65.Ud Entanglement and quantum nonlocality (e.g. EPR paradox, Bell's inequalities, GHZ states, etc.) – 03.67.Lx Quantum computation – 07.60.Ly Interferometers

1 Introduction

Entanglement is a quintessential property of quantum mechanics that sets it apart from any classical physical theory. An important feature of entanglement is that it gives rise to correlations that cannot be explained by any local realistic description of quantum mechanics. The idea of non-local correlation among remote particles was originally broached in a classic paper on the incompleteness of quantum mechanics by Einstein, Podolsky and Rosen [1]. In this paper, Einstein and his co-workers proposed a Gedenken experiment involving two entangled particles which showed that quantum mechanics cannot in all situations be a complete description of physical reality. This idea was subsequently conceptualized in a seminal paper by Schrödinger [2] and revisited in a subsequent work by Bell [3]. Incidentally, these particles (EPR pairs) have now found wide applications in the area of quantum information theory.

An entangled state is state of a composite system that cannot be separated into product states in terms of the subsystems. For a bipartite pure state, the degree of entanglement can be found from the Schmidt numbers. For a mixed state, there is the Peres-Horodecki theorem concerning partial transposition which can be used to determine if a state is entangled or separable provided the dimension is low, specifically 2×2 or 2×3 systems. Indeed, in recent years, quantum entanglement has become such an important physical resource for quantum communication and information processing that it has found wide applications in processes like quantum teleportation [4,5], superdense coding [6], quantum key distribution [7] and telecoloning [8].

Besides quantum entanglement, quantum mechanics harbors another surprisingly elegant idea. This idea was hatched from fact that a quantum state acquires a purely geometrical phase under adiabatic evolution. This phase depends geometrically on the area covered by the evolution of the system but it does not depend on how the motion is performed. Indeed, from a historical perspective, the concept of geometric phase was originally introduced by Pancharatnam [9] in the context of interference between light waves in distinct states of polarization. It was subsequently rediscovered by Berry [10] for quantal systems undergoing cyclic adiabatic evolution. This Berry phase has since then been linked to the notion of parallel transport [11] and formulated elegantly as well as rigorously using the language of differential geometry.

Quantum entanglement and geometric phase have been extensively explored for physical systems with two states or two levels (qubits) [12,13]. Some common examples of two-state system are atoms [14,15] with spin $S_z = \pm\frac{1}{2}$ or photons [16] where one represents the system in terms of the two polarized states. Experimentally, quantum interference with a photon beam can be generated from a parametric down-converter and employed in a large number of experiments [17–19] to verify and probe the intriguing properties of quantum entanglement and quantum geometric phase.

While it is generally possible to study physical systems with two states or two levels both theoretically and experimentally, the extension to higher dimensions (more

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states) is often fraught with experimental challenges and difficulty. For instance, in an interacting many-particle system, while one may acknowledge the existence of entanglement, there is still the question of how to characterize the degree of entanglement [20]. Indeed, some of the most challenging and interesting problems in quantum mechanics concern many-body systems with strong quantum fluctuations. Moreover, there are many possible theoretical insights by studying many body problems that could potentially enhance our understanding of strongly-coupled systems for applications of quantum information theory.

Recently, there have also been increasing interests in the study of entangled state of spin-s objects ($s > \frac{1}{2}$), which apart from its fundamental interest [21–24], are clearly of some practical interest in the applications to quantum information such as quantum cryptography [25,26]. Experimental violation of a spin-1 Bell inequality has been reported using polarization entangled four spin-1 photon state [27]. However, the experimental generation of spin-1 photon states using two correlated photons (bi-photon) prepared via spontaneous parametric down-conversion techniques are inherently composed of massless particles.

Experimental observation of bosonic Mott-Hubbard transition in Rb atoms in higher spin particles have been reported in reference [28]. Following this observation, Yip [29] considered spin-1 bosons trapped in an optical lattice and argued that the ground state should be a dimer phase. Other related studies include the investigation of rotating spin-1 Bose cluster [30], experimental observation of multiparticle states of quasi-one dimensional spin-1 chain [31] and the solution of spin-1 anti-ferromagnetic chain doped with mobile spin-1/2 carriers [32]. Since entanglement in optical lattice could be used as a possible physical resource for quantum computation, it is therefore interesting to consider entanglement for massive spin-1 particles.

In this paper, we focus on the study of quantum entanglement and the quantum geometric phase for massive spin-1 particles in many particle system. In Section 2, we review and describe spin-1 representation for massive spin-1 particles and the spin-spin interaction between the transverse and the longitudinal states. In Section 3, we work out the geometric phase and entanglement for a two spin-1 particle system and a linked equation between the geometric phase and entanglement is studied, while in Section 4, we extend the calculation to 3 and 4 spin-1 particles. In Section 5, we briefly discuss the situation for N spin-1 particles. In real systems noise and decoherence are a big problem. The process limits the ability to maintain pure quantum states and quantum information. Therefore, in Section 6, the interaction of the N-particle system with its environment is briefly discussed.

2 Spin-1 representation

A massive spin-1 particle has three distinct basis states. It is convenient to use a representation in order to describe the components of the angular momentum of a spin-1 particle. Three spin matrices may be expressed as [33],

$$S^x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \; S^y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix},$$
$$S^z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{1}$$

in natural units ($\hbar = 1$). These matrices obey the commutation relations of the angular momentum.

Moreover, we know that the transverse and longitudinal polarization vectors are related to the eigenvectors of the helicity operator $\mathbf{S} \cdot \mathbf{p}/|\mathbf{p}|$ where $\mathbf{p}$ is momentum of the spin-1 particle. A convenient choice is that $\mathbf{p}$ points along the positive z-axis. With the same reference system we could choose the eigenvectors of the spin-1 particle as

$$|1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, \; |2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}, \; |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{2}$$

and

$$S^z|1\rangle = |1\rangle, \; S^z|2\rangle = -|2\rangle, \; S^z|3\rangle = 0. \tag{3}$$

The polarized vectors $|1\rangle$ (spin up) and $|2\rangle$ (spin down) represent states $|T\rangle$ of transverse polarization, while $|3\rangle$ (spin 0) represents longitudinal polarization $|L\rangle$. With the help of the spin matrices in equation (1), we can construct raising and lowering operators defined by

$$S^+ = \frac{1}{\sqrt{2}}(S^x + iS^y), \; S^- = \frac{1}{\sqrt{2}}(S^x - iS^y), \tag{4}$$

acting on the states of spin-1 particle.

We may now construct a spin-spin interaction between the transverse and the longitudinal states in an isolated quantum system consisting of N spin-1 particles using the Hamiltonian

$$\hat{\mathcal{H}} = \frac{\lambda}{2} \sum_{n=1}^{N} \left(S_n^{L+} S_{n+1}^{T-} + S_n^{T-} S_{n+1}^{L+} + S_n^{T+} S_{n+1}^{L-} \right.$$
$$\left. + S_n^{L-} S_{n+1}^{T+} + S_n^{Lz} S_{n+1}^{Tz} + S_n^{Tz} S_{n+1}^{Lz} \right), \tag{5}$$

where $S_n^{L\pm}|L\rangle = S_n^{\pm}|L\rangle$, $S_n^{L\pm}|T\rangle = 0$, $S_n^{T\pm}|L\rangle = 0$, $S_n^{T\pm}|T\rangle = S_n^{\pm}|T\rangle$, $S_n^{Lz}|L\rangle = S_n^{z}|L\rangle$, $S_n^{Lz}|T\rangle = 0$, $S_n^{Tz}|L\rangle = 0$, and $S_n^{Tz}|T\rangle = S_n^{z}|T\rangle$. A similar action to the operators, $S_{n+1}^{L\pm}$, $S_{n+1}^{T\pm}$, S_{n+1}^{Lz} and S_{n+1}^{Tz}, Moreover, we have imposed a periodic boundary condition, $S_{N+1}^{\pm,z} = S_1^{\pm,z}$, and where λ is the strength of the interaction.

The Hamiltonian operator (5) in the case of two spin$-\frac{1}{2}$ particles has been considered in an implementation of holonomic quantum computation using nuclear magnetic resonance where λ decreases with the spatial distance between particles [15]. It is therefore interesting to consider the degree of entanglement and the geometric phase for the spin-1 particle particles.

The eigenvalue problem of equation (5) can be solved exactly using the Jordan-Wigner transformation [34]. To obtain the time-evolution of a state under the Hamiltonian, we define a N spin-1 ground state as a linear superposition of $N-1$ spin $S_3 = 0$ states and one spin $S_3 = 1$ state,

$$|k\rangle = \sum_{n=1}^{N} a_{k,n} S_n^+ |3\rangle^{\otimes N}, \tag{6}$$

where $k = 1$ for ground state; or $N-1$ spin $S_3 = 0$ states and one spin $S_3 = -1$ state,

$$|l\rangle = \sum_{n=1}^{N} b_{l,n} S_n^- |3\rangle^{\otimes N}, \tag{7}$$

where $l = 1$ for ground state.

Using the properties of the defined raising and lowering operators and taking expression (6) as the eigenstate of the Hamiltonian, we have,

$$\begin{aligned}\hat{\mathcal{H}}|k\rangle &= \sum_{n=1}^{N} a_{k,n} \hat{\mathcal{H}} S_n^+ |3\rangle^{\otimes N} \\ &= \frac{\lambda}{2} \sum_{n,m} \left(S_m^{L+}\delta_{m+1,n} + S_{m+1}^{L+}\delta_{m,n}\right)|3\rangle^{\otimes N} \\ &= \frac{\lambda}{2} \sum_{n=1}^{N} \left(a_{k,n+1} + a_{k,n-1}\right) S_n^+ |3\rangle^{\otimes N} \\ &= E_k |k\rangle = E_k \sum_{n=1}^{N} a_{k,n} S_n^+ |3\rangle^{\otimes N}. \end{aligned} \tag{8}$$

From the above equation, the coefficients $a_{k,n}$ are found to satisfy a recursion relation,

$$\frac{\lambda}{2}\left(a_{k,n+1} + a_{k,n-1}\right) = E_k a_{k,n}. \tag{9}$$

Applying a periodic boundary condition to equation (9), we have

$$a_{k,n} = \exp\left(i\frac{(2\pi k + \phi)n}{N}\right), \tag{10}$$

$$E_k = \lambda \cos\left(\frac{(2\pi k + \phi)}{N}\right). \tag{11}$$

It should be noted that in equations (10) and (11), we have included the phase, $0 \le \phi \le 2\pi$, unlike the case of a previous work [34] on spin-1/2 particles in which the phase is left out. The eigenvectors are given by

$$|k\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \exp\left(i\frac{(2\pi k + \phi)n}{N}\right) S_n^+ |3\rangle^{\otimes N}, \tag{12}$$

and satisfy the normalization condition. It is noted that these eigenstates can be considered as generalized Greenberger-Horme-Zeilinger (GHZ) states [35,36] with arbitrary phases $\theta_i (i = 1, 2, ..., N)$ for any N-particle system. Indeed any particle pair in the above state after tracing out the rest of the particles is maximally mixed.

An initial state is needed to describe time-evolution of N spin-1 particle system. In terms of equation (12), we define the initial state of N spin-1 particles as

$$|\psi_N(0)\rangle = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \exp\left(-i\frac{(2\pi k + \phi)}{N}\right)|k\rangle, \tag{13}$$

and the state vector at time t may be obtained by Schrödinger equation

$$|\psi_N(t)\rangle = \sum_{n=1}^{N} C(n, N, t) S_n^+ |3\rangle^{\otimes N}, \tag{14}$$

where the coefficient is given by

$$C(n, N, t) = \frac{1}{N}\sum_{k=1}^{N} \exp\left(i\frac{(2\pi k + \phi)(n-1)}{N} - i\lambda t \cos\frac{(2\pi k + \phi)}{N}\right). \tag{15}$$

3 Two spin-1 particles system

Although we are interested in the geometric phase and the degree of entanglement for N particles, it is instructive to consider the simplest case of two spin-1 particles. For two spin-1 particles, we find

$$C(1, 2, t) = \cos\lambda' t,\ C(2, 2, t) = -ie^{i\frac{\phi}{2}} \sin\lambda' t, \tag{16}$$

where $\lambda' = \lambda \cos\frac{\phi}{2}$ is introduced as a result of the renormalization for the coupling constant of the spin-spin interaction. $0 < \lambda' t < \pi$ and $0 < \phi < 2\pi$ correspond to the polar and azimuthal angles respectively. Moreover, we have appropriately rescaled the interaction strength. Note also that $|C(1, 2, t)|^2 + |C(2, 2, t)|^2 = 1$.

Inserting equations (16) into (14), we find that the wavefunction of two spin-1 particle system is given by

$$|\psi_2(\lambda', t)\rangle = -\cos\lambda' t|13\rangle + ie^{i\frac{\phi}{2}} \sin\lambda' t|31\rangle. \tag{17}$$

When $\lambda' t = 2n\pi + \pi/4$, $|\psi_2(\lambda', t)\rangle$ is maximally entangled state. At these points, the corresponding phases of the generalized GHZ state (called as W state) are $\theta_1 = \pi$ and $\theta_2 = (\pi + \phi))/2$.

Similarly, one has

$$|\varphi_2(\lambda', t)\rangle = \cos\lambda' t|23\rangle - ie^{i\frac{\phi}{2}} \sin\lambda' t|32\rangle, \tag{18}$$

we see from equations (17) and (18) that the Hamiltonian (5) cannot distinguish the interaction between the spin up state and the spin down state. When $\lambda' t = 2n\pi (n = 1, 2...)$, $|\psi_2(\lambda', t)\rangle$ and $|\varphi_2(\lambda', t)\rangle$ undergo cyclic evolutions. Such as $|\psi_2(0)\rangle = |\psi_2(\tau)\rangle$. The cyclic condition

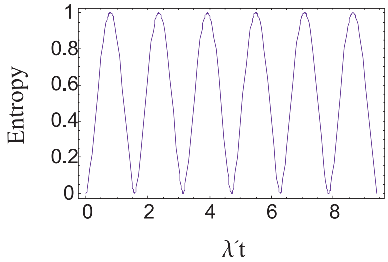

Fig. 1. Entropy of the subsystem of a two spin-1 particle system as a function of $\lambda' t$ is shown. The state of the two particles evolves cyclically from a pure state to a maximally mixed one. When $\lambda' t = \pi/4, 3\pi/4, ...$, Entropy has a maximum value, this corresponds to a maximally entangled state. When $\lambda' t = 0, \pi/2, \pi, ...$, the system is in pure state.

$\tau = 2n\pi/\lambda' = 2n\pi/(\lambda \cos\frac{\phi}{2})$ for the motion is dependent on the phase ϕ.

From equations (17) and (18), we find

$$\langle \psi_2(\lambda', t)|\frac{d}{dt}|\psi_2(\lambda', t)\rangle = 0,$$
$$\langle \varphi_2(\lambda', t)|\frac{d}{dt}|\varphi_2(\lambda', t)\rangle = 0, \tag{19}$$

which is the parallel-transport condition of the two spin-1 particle system [11]. This condition has its origin in formalism of Schrödinger equation and it has a purely geometric [10,11] origin.

For a bipartite system, we can consider the entropy of a subsystem as a measure of entanglement. To do this, we first calculate density matrix of the system

$$\rho_{AB} = |\psi_2(\lambda', t)\rangle_{AB} \times {}_{AB}\langle \psi_2(\lambda', t)|, \tag{20}$$

and take the partial trace so that the mixed state of the subsystem A is given by

$$\rho_A = \mathrm{Tr}_B(\rho_{AB}) = \begin{pmatrix} \cos^2\lambda' t/2 & -i\cos^2\lambda' t/2 & 0 \\ i\cos^2\lambda' t/2 & \cos^2\lambda' t/2 & 0 \\ 0 & 0 & \sin^2\lambda' t \end{pmatrix}. \tag{21}$$

In order to describe the degree of impurity of mixed states, ρ_A, we compute the von Neumann entropy [37] of the subsystem. The entropy is given by

$$S(\rho_A) = -\left[\cos^2(\lambda' t)\right]\log_2\left[\cos^2(\lambda' t)\right] - \left[\sin^2(\lambda' t)\right]\log_2\left[\sin^2(\lambda' t)\right]. \tag{22}$$

The entropy varies from 0 to its maximum value ($\log_2 2$), and the state of the system of two spin-1 particles evolves from a pure state to a maximally mixed one. The entanglement of two spin-1 particles is time dependent, as shown in Figure 1. It exhibits a cyclic behavior and the cyclic behavior is the same as the behavior of the wave function.

The entanglement of two-particle system may be understood by the Schmidt decomposition, where the Schmidt number of $|\psi_2(\lambda', t)\rangle$ is two when $\lambda' t \neq 0, \pi/2, \pi, ...$ This means that the system is entangled state. when $\lambda' t = 0, \pi/2, \pi, ...$, the Schmidt number is changed to one. In this case, the system is unentangled state [38].

Another way to describe the entanglement is to calculate the concurrence by taking the basis $\{|11\rangle, |13\rangle, |31\rangle, |33\rangle\}$. We find that the concurrence is $|\sin 2\lambda' t|$. When $\lambda' t = \pi/4, 3\pi/4, ...$, $|\sin 2\lambda' t| = 1$ corresponds to a maximally entangled state. When $\lambda' t = 0, \pi/2, \pi, ...$, $|\sin 2\lambda' t| = 0$ corresponds to a unentangled state. Therefore, the description of the entropy, the Schmidt decomposition and the concurrence to the entanglement of the two-particle system is consistent.

It is known that the state $|\psi\rangle$ in a complex Hilbert space $\mathcal{H}$ is physically indistinguishable from the state $|\psi'\rangle = e^{i\chi}|\psi\rangle$. In other words, the initial and final states should be found along the same ray in $\mathcal{H}$, but may be related to each other by a phase [39]. Under this condition, the quantity,

$$\arg\langle\psi'(t_1)|\psi'(t_2)\rangle + i\int_{t_1}^{t_2}\langle\psi'(t)|d|\psi'(t)\rangle = \arg\langle\psi(t_1)|\psi(t_2)\rangle + i\int_{t_1}^{t_2}\langle\psi(t)|d|\psi(t)\rangle, \tag{23}$$

is invariant. Moreover, this functional is reparameterization invariant [40]. When a quantum system undergoes a cyclic evolution, the first term on the left and on the right in equation (23) only contributes a factor 2π.

By considering the projective space, $\mathcal{P}$, in which vectors are grouped under equivalence classes $|\psi\rangle \sim re^{i\chi}|\psi\rangle$ for any $r > 0$ and real χ, the associated projection map is

$$\Pi:\ \mathcal{H} \to \mathcal{P}$$
$$|\psi\rangle \to \Pi(|\psi\rangle) = \left\{|\psi'\rangle : |\psi'\rangle = re^{i\chi}|\psi\rangle\right\}, \tag{24}$$

and the ket representing the system state traces out a path, $\mathcal{C}$: $[0, \tau] \to \mathcal{H}$, where $\Pi(\mathcal{C})$ is closed curve in $\mathcal{P}$. For each point $|\psi\rangle$ on $\mathcal{C}$, we can choose a smoothly varying representative $|\psi\rangle$ from $\Pi(|\psi(t)\rangle$ in such a way that $|\psi(0)\rangle = |\psi(\tau)\rangle$. $|\psi\rangle$ represents the evolved state of the system and satisfies the condition (19) for parallel transport.

From equations (17) and (18), we know that ϕ is a parameter controlling the renormalized Hamiltonian. If ϕ traces a closed loop in the parameter space, the geometric (Berry) phase of the system can then be written in terms of ϕ as

$$\gamma_g(\mathcal{C}) = -\frac{\pi}{2}[1 - \cos(2\lambda' t)]. \tag{25}$$

Like the case of the entanglement, the geometric phase of two spin-1 particles is also time dependent. It is a cyclic (see Fig. 2) with a period which is half the time needed for the evolution of the two spin-1 particle system. This is that the geometric phase depends only on the area covered by the evolution of the system. For $\lambda' t = n\pi$, the phase vanishes and for $\lambda' t = (n \pm \frac{1}{2})\pi$, the geometric phase is $-\pi$.

From equations (22) and (25) and Figures 1–2, we see that the entanglement measured using the entropy of one

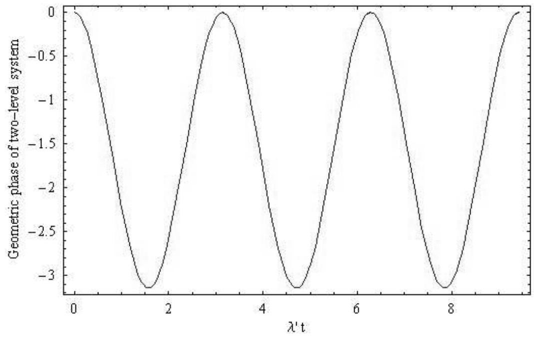

Fig. 2. Geometric phase γ_g of two-level system as a function of $\lambda' t$ is shown. It is time-dependent and exhibits a cyclic phase. It is shown that the geometric phase changes periodically as the state of two particles alternates between pure state to mixed state.

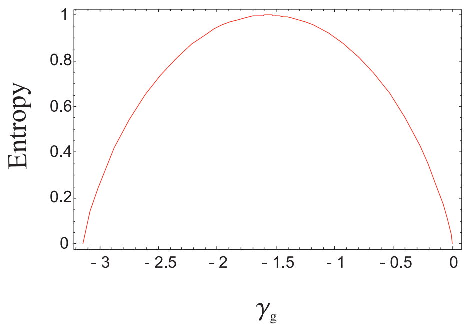

Fig. 3. Entropy of the subsystem of the spin-1 two-particle system as a function of the geometric phase is shown. When the geometric phase change from $-\pi$ to $-\pi/2$ to 0, the entropy as a measure of entanglement goes from 0 to a maximum to 0.

of the subsystems of the two spin-1 particle system is related to its geometric phase. Inserting equation (25) into equation (22), we find

$$S = \frac{\gamma_g}{\pi}\log_2\frac{-\gamma_g}{\pi} - \left(1+\frac{\gamma_g}{\pi}\right)\log_2\left(1+\frac{\gamma_g}{\pi}\right). \tag{26}$$

The relation between the entropy and geometric phase is shown in Figure 3. It is obvious there is a symmetry between two sides of the point $\gamma_g = -\pi/2$. At the point of $\gamma_g = -\pi/2$, the entropy is equal to one. This means that the system is a maximally entangled state. At the minimum and maximum points of $\gamma_g = -\pi, 0$, the entropy is zero. This means that the system is a unentangled state. Therefore, the cyclic geometric phase includes all information of the two-particle entanglement. Equation (26) may be interesting in application of quantum information.

4 Three and four spin-1 particles systems

In recent years, three-qubit entangled states have been investigated by a number of authors [41–44]. They have also been shown to have certain advantages over the two-particle Bell states in their applications to dense coding, teleportation and quantum cloning. Moreover, it is also common in atomic and molecular physics to consider the cases of N ($N > 2$) spin-1 particles and their resonances.

In this section, we extend the previous section to a system of three and four spin-1 particles under the same type of spin-spin interaction. Using equation (15) by setting $N = 3$, one has

$$C(1,3,t) = \frac{1}{3}\left\{2\exp\left(i\frac{1}{2}\lambda_1 t\right)\cos\left(\frac{\sqrt{3}}{2}\lambda_2 t\right) + \exp\left(-i\lambda_1 t\right)\right\}, \tag{27}$$

where $\lambda_1 = \lambda\cos\frac{\phi}{3}$ and $\lambda_2 = \lambda\sin\frac{\phi}{3}$ are rescaled coupling constants, and

$$C(2,3,t) = -\frac{1}{3}e^{i\frac{\phi}{3}}\left\{\exp\left(i\frac{1}{2}\lambda_1 t\right)\left[\cos\left(\frac{\sqrt{3}}{2}\lambda_2 t\right) + \sqrt{3}\sin\left(\frac{\sqrt{3}}{2}\lambda_2 t\right)\right] - \exp\left(-i\lambda_1 t\right)\right\}, \tag{28}$$

$$C(3,3,t) = \frac{1}{3}e^{i\frac{2\phi}{3}}\left\{\exp\left(i\frac{1}{2}\lambda_1 t\right)\left[-\cos\left(\frac{\sqrt{3}}{2}\lambda_2 t\right) + \sqrt{3}\sin\left(\frac{\sqrt{3}}{2}\lambda_2 t\right)\right] + \exp\left(-i\lambda_1 t\right)\right\}. \tag{29}$$

Moreover, the analytic expressions for the associated probabilities are

$$P(1,3,t) = |C(1,3,t)|^2 = \frac{1}{9}\left[1 + 4\cos^2\left(\frac{\sqrt{3}}{2}\lambda_2 t\right) + 4\cos\left(\frac{3}{2}\lambda_1 t\right)\cos\left(\frac{\sqrt{3}}{2}\lambda_2 t\right)\right], \tag{30}$$

$$\begin{aligned}P(2,3,t) &= |C(2,3,t)|^2\\ &= \frac{1}{9}\left(1 + \left[\cos\left(\frac{\sqrt{3}}{2}\lambda_2 t\right) + \sqrt{3}\sin\left(\frac{\sqrt{3}}{2}\lambda_2 t\right)\right]^2\right.\\ &\quad\left. - 2\cos\left(\frac{3}{2}\lambda_1 t\right)\left[\cos\left(\frac{\sqrt{3}}{2}\lambda_2 t\right) + \sqrt{3}\sin\left(\frac{\sqrt{3}}{2}\lambda_2 t\right)\right]\right),\end{aligned} \tag{31}$$

$$\begin{aligned}P(3,3,t) &= |C(3,3,t)|^2\\ &= \frac{1}{9}\left(1 + \left[-\cos\left(\frac{\sqrt{3}}{2}\lambda_2 t\right) + \sqrt{3}\sin\left(\frac{\sqrt{3}}{2}\lambda_2 t\right)\right]^2\right.\\ &\quad\left. + 2\cos\left(\frac{3}{2}\lambda_1 t\right)\left[-\cos\left(\frac{\sqrt{3}}{2}\lambda_2 t\right) + \sqrt{3}\sin\left(\frac{\sqrt{3}}{2}\lambda_2 t\right)\right]\right).\end{aligned} \tag{32}$$

Note that it is easy to verify that $P(1,3,t)+P(2,3,t)+P(3,3,t)=1$.

From equations (27)–(29), we see that we need two parameters to describe the motion of the three spin-1 particles. By a suitable rescaling in the interaction between the three spin-1 particles, these parameters are denoted by $\lambda_1 t$ and $\lambda_2 t$. From equations (30)–(32), we see that they can undergo very complicating cyclic motion.

We see if $\cos(\frac{3}{2}\lambda_1 t)=\pm\frac{1}{2}$ and $\cos(\frac{\sqrt{3}}{2}\lambda_2 t)=\pm\frac{1}{2}$, the probabilities are the same. This implies, at some points of $\lambda_1 t=\frac{2\pi}{9}+\frac{4n\pi}{3}$ and $\lambda_2 t=\frac{2\pi}{3\sqrt{3}}+\frac{4n\pi}{\sqrt{3}}$ or $\lambda_1 t=\frac{4\pi}{9}+\frac{4n\pi}{3}$ and $\lambda_2 t=\frac{4\pi}{3\sqrt{3}}+\frac{4n\pi}{\sqrt{3}}$, the corresponding state vectors are the W states, such as

$$|\psi_W\rangle=\frac{1}{\sqrt{3}}\left(e^{i\frac{\pi}{6}}|133\rangle-ie^{i\frac{\phi}{3}}|313\rangle+e^{i\left(\frac{2}{3}\phi+\frac{\pi}{6}\right)}|331\rangle\right), \tag{33}$$

or

$$|\psi_W\rangle=\frac{1}{\sqrt{3}}\left(e^{i\frac{2\pi}{3}}|133\rangle-e^{i\frac{\phi-\pi}{3}}|313\rangle+ie^{i\frac{2}{3}\phi}|331\rangle\right). \tag{34}$$

Three-particle entangled W state, which is inequivalent to the GHZ state under stochastic local operations and classical communication, is robust in that it remains entangled even after any one of the three particles is traced out [40–43].

Inserting equations (27)–(29) into (14), we get

$$\begin{aligned}|\psi_3(\lambda_1,\lambda_2,t)\rangle=&-C(1,3,t)|133\rangle\\&-C(2,3,t)|313\rangle-C(3,3,t)|331\rangle,\end{aligned} \tag{35}$$

a similar equation for $|\phi_3\rangle$ can be obtained.

There is no generically acceptable measure for degree of entanglement for a tripartite system consisting of three spin-1 particles. One method involves looking at the bipartite entanglement for neighboring sites. Thus, we could compute the concurrence or simply the Von Neumann entropy of the subsystem, such as

$$\rho_{AB}=\mathrm{Tr}_C(\rho_{ABC}). \tag{36}$$

It is also interesting to compare the entropy of the bipartite system with the entropy at each site by calculating the entropy of the density matrix of the system,

$$\begin{aligned}\rho_A&=\mathrm{Tr}_{BC}(\rho_{ABC})\\&=\begin{pmatrix}|C(1,3,t)|^2/2 & -i|C(1,3,t|^2/2 & 0\\ i|C(1,3,t)|^2/2 & |C(1,3,t)|^2/2 & 0\\ 0 & 0 & 1-|C(1,3,t)|^2\end{pmatrix}.\end{aligned} \tag{37}$$

The entropy function may be expressed by

$$\begin{aligned}S(\rho_A)=&-\mathrm{Tr}\left(\rho\log_2\rho\right)\\=&-|C(1,3,t)|^2\log_2|C(1,3,t)|^2\\&-\left(|C(2,3,t)|^2+|C(3,3,t)|^2\right)\\&\times\log_2\left(|C(2,3,t)|^2+|C(3,3,t)|^2\right).\end{aligned} \tag{38}$$

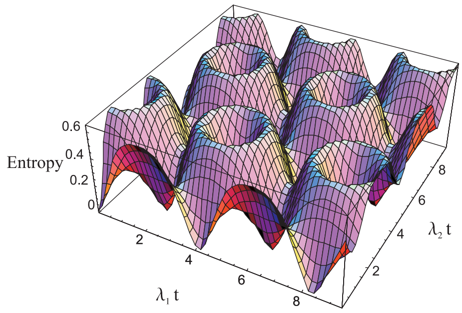

Fig. 4. The entropy for the subsystem of two neighboring sites, AB, as a function of $\lambda_1 t$ and $\lambda_2 t$. Note the cyclic variation in the entropy as the parameters of the system changes. In general it is not possible to attain the maximum value of unity. Notice that ρ_{AB} is exactly the same as ρ_C.

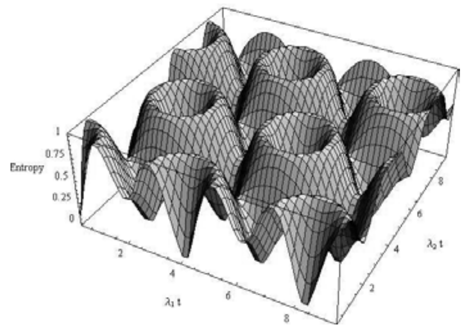

Fig. 5. The entropy of the subsystem of two neighboring sites, BC, as a function of $\lambda_1 t$ and $\lambda_2 t$. It is shown that the entropy varies cyclically from 0 to 1 as the state of three particles evolves from a pure state to a maximally mixed one. Notice that the plot is exactly the same as the one for the entropy of the bipartite system of a single site A.

The Von Neumann entropies of the bipartite system, AB and BC, are plotted as a function of $\lambda_1 t$ and $\lambda_2 t$ in Figures 4 and 5 respectively. The entropies of the subsystems of three spin-1 particles (see Figs. 4 and 5) are naturally more complicating than the case of two spin-1 particles (see Fig. 1). In particular, they are now parameterized by two time-dependent parameters, $\lambda_1 t$ and the $\lambda_2 t$ (see Figs. 4 and 5), in contrast to the case of the two spin-1 particles system (see Fig. 1) which can be parameterized by a single time-dependent variable. However the general cyclical nature of the variation is still present. Moreover, the entropies for the subsystem, ρ_{AB} and ρ_{BC}, are exactly the same as the entropies for ρ_C and ρ_A, respectively. This may be understood by symmetry of the wave function (35) of the three-particle system in which three basis are $|133\rangle$, $|313\rangle$, and $|331\rangle$ respectively. Therefore $S(\rho_A)$ can be taken as a measurement of entanglement for the three particle system in our model.

The concurrence for the system may be calculated under the same basis, such as $\{|11\rangle,|13\rangle,|31\rangle,|33\rangle\}$. They are $2\sqrt{P(1,3,t)P(2,3,t)}$, $2\sqrt{P(2,3,t)P(3,3,t)}$ and $2\sqrt{P(3,3,t)P(1,3,t)}$, which are associated to the probabilities of the system.

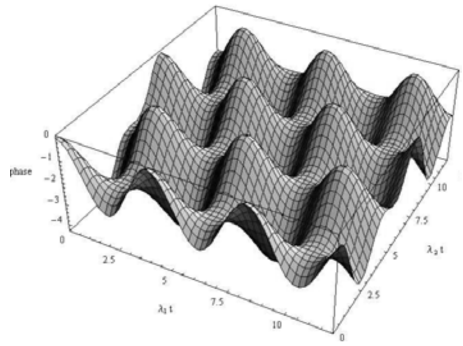

Fig. 6. Three-dimensional plot of geometric phase of three spin-1 particles. As the state of three particles evolves from a pure state to a mixed state, the geometric phase is cyclically changed. The cyclic geometric phase returns a memory of its motion.

In the Schmidt decomposition of the three-particle system, the Schmidt number is three in general case. This means that the system is a entangled state [38]. However, when $P(1,3,t) = P(2,3,t) = 0$ and $P(3,3,t) = 1$, $P(2,3,t) = P(3,3,t) = 0$ and $P(1,3,t) = 1$ or $P(3,3,t) = P(1,3,t) = 0$ and $P(2,3,t) = 1$, the Schmidt number of the system is changed to one. Thus, the system is a unentangled state. It is obvious that the conclusions are consistent with description of the entropy $S(\rho_A)$ and the concurrence to the entanglement.

From equation (35), we see that the wave functions is cyclic under a suitable choice of parameters, such as $\frac{1}{2}\lambda_1 t = 2n\pi$, $\frac{\sqrt{3}}{2}\lambda_2 t = 2m\pi; (n, m = 1, 2, ...)$, which is satisfied for $\sqrt{3}\tan\frac{\phi}{3} = \frac{m}{n}$. Thus, when the system of three spin-1 particles undergoes such a cyclic evolution, the system will also trace a closed path in the projective space. Moreover, a direct calculation shows that they fulfill the condition of parallel transport. Thus, the geometric phase of the cyclic motion is given by

$$\begin{aligned}\gamma_g =& i\oint_C d\phi \left\langle \psi_3(\lambda_1,\lambda_2,t)\left|\frac{d}{d\phi}\right|\psi_3(\lambda_1,\lambda_2,t)\right\rangle \\ =& -\frac{2\pi}{27}\left[6 + 6\sin^2\left(\frac{\sqrt{3}}{2}\lambda_2 t\right) - \sqrt{3}\sin\left(\sqrt{3}\lambda_2 t\right)\right. \\ & - 6\cos\left(\frac{3}{2}\lambda_1 t\right)\cos\left(\frac{\sqrt{3}}{2}\lambda_2 t\right) \\ & \left. +2\sqrt{3}\cos\left(\frac{3}{2}\lambda_1 t\right)\sin\left(\frac{\sqrt{3}}{2}\lambda_2 t\right)\right]. \end{aligned} \tag{39}$$

The cyclic geometric phase of three spin-1 particle system is plotted in Figure 6. Compared to the two spin-1 particle system (see Fig. 2), this is a more intricate plot with two parameters. From equations (30)–(32) and Figure 6, the geometric phase depends on the probability functions. This means that the geometric phase is changed from $-\pi$ to zero according to the probabilities of the system. In the form of the geometric phase factor, therefore, we know that the wave function of the three-particle system may retain a memory of its motion. This phase factor can be measured by interfering the wave function with another coherent wave function enabling one to discern whether or not the system has undergone an evolution.

We next extend our study to four spin-1 particles. It is noteworthy that recent experimental violation of a spin-1 Bell inequality has been reported using polarization entangled four spin-1 particle state produced by pulsed parameter down conversion. In this case, the coefficients from equation (15) may be written as

$$C(1,4,t) = \frac{1}{2}\left[\cos(\lambda_4 t) + \cos(\lambda_3 t)\right], \tag{40}$$

$$C(2,4,t) = -\frac{1}{2}e^{i\frac{\phi}{4}}[\sin(\lambda_4 t) + i\sin(\lambda_3 t)], \tag{41}$$

$$C(3,4,t) = -\frac{1}{2}e^{i\frac{\phi}{2}}[\cos(\lambda_4 t) - \cos(\lambda_3 t)], \tag{42}$$

$$C(4,4,t) = \frac{1}{2}e^{i\frac{3\phi}{4}}[\sin(\lambda_4 t) - i\sin(\lambda_3 t)], \tag{43}$$

where $\lambda_3 = \lambda\cos\frac{\phi}{4}$ and $\lambda_4 = \lambda\sin\frac{\phi}{4}$ are renormalized coupling constants. And the corresponding probabilities,

$$\begin{aligned} P(1,4,t) &= |C(1,4,t)|^2 \\ &= \frac{1}{4}\left[\cos(\lambda_4 t) + \cos(\lambda_3 t)\right]^2, \end{aligned} \tag{44}$$

$$\begin{aligned} P(2,4,t) &= P(4,4,t) \\ &= \frac{1}{4}\left[\sin^2(\lambda_4 t) + \sin^2(\lambda_3 t)\right], \end{aligned} \tag{45}$$

$$\begin{aligned} P(3,4,t) &= |C(3,4,t)|^2 \\ &= \frac{1}{4}\left[\cos(\lambda_4 t) - \cos(\lambda_3 t)\right]^2. \end{aligned} \tag{46}$$

The probabilities in equations (44)–(46) are the cyclic functions of two-parameters, namely $\lambda_3 t$ and $\lambda_4 t$.

When $\cos(\lambda_3 t) = \pm 1$ and $\cos(\lambda_4 t) = 0$ (or $\cos(\lambda_3 t) = 0$ and $\cos(\lambda_4 t) = \pm 1$) are satisfied, the probabilities are same. Explicitly at some points of $\lambda_3 t = n\pi$ and $\lambda_4 t = (n + \frac{1}{2})\pi$ (or $\lambda_3 t = (n + \frac{1}{2})\pi$ and $\lambda_4 t = n\pi$), the corresponding state vectors are

$$\begin{aligned} |\psi_W\rangle =& \frac{1}{\sqrt{4}}\Big(|1333\rangle - e^{i\frac{\phi}{4}}|3133\rangle \\ & + e^{i\frac{\phi}{2}}|3313\rangle + e^{i\frac{3\phi}{4}}|3331\rangle\Big), \end{aligned} \tag{47}$$

or,

$$\begin{aligned} |\psi_W\rangle =& \frac{1}{\sqrt{4}}\Big(|1333\rangle - ie^{i\frac{\phi}{4}}|3133\rangle \\ & - e^{i\frac{\phi}{2}}|3313\rangle - ie^{i\frac{3\phi}{4}}|3331\rangle\Big), \end{aligned} \tag{48}$$

which are the W states. The GHZ and W states exhibit very different properties when subjected to physical processes like state loss, or white noise [40–43].

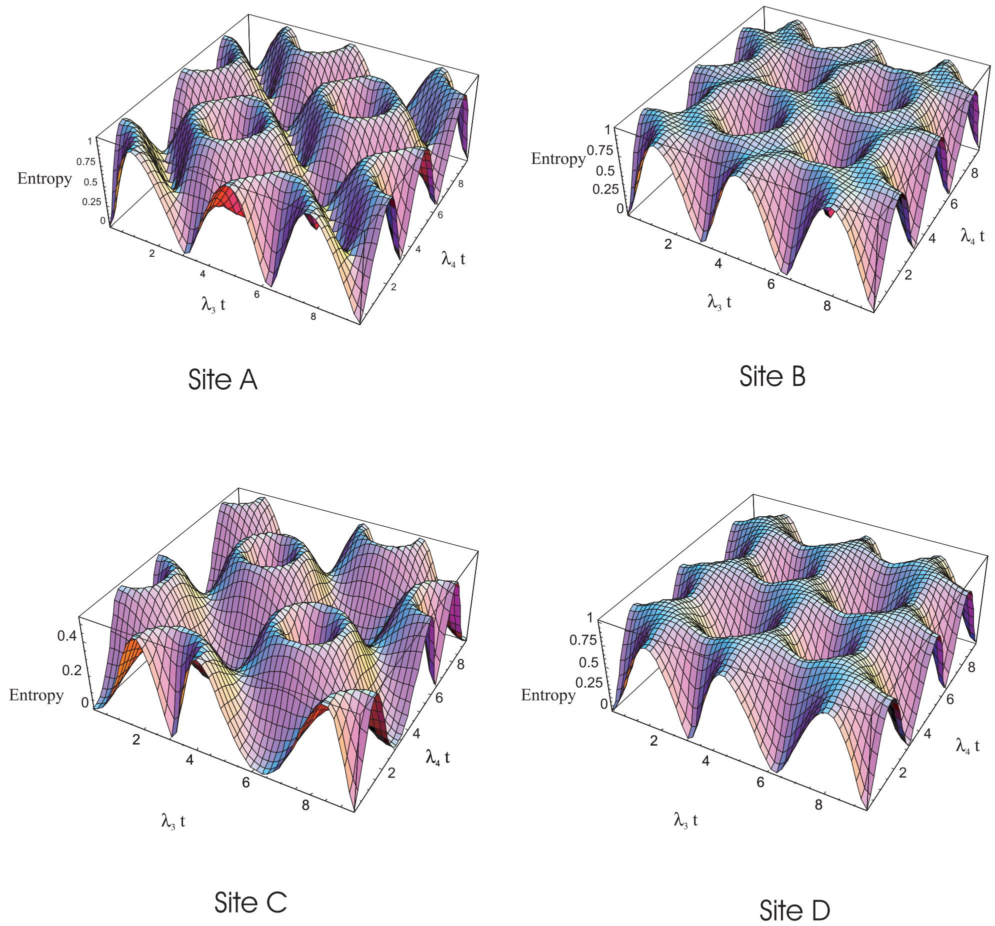

Fig. 7. Plots of entropy for the reduced density matrix of a single site for a four-spin-1 particle system.

The wave functions of the four spin-1 particle states are

$$|\psi_4(\lambda_3, \lambda_4, t)\rangle = - C(1,4,t)|1333\rangle - C(2,4,t)|3133\rangle - C(3,4,t)|3313\rangle - C(4,4,t)|3331\rangle, \quad (49)$$

and $|\phi_4(\lambda_3, \lambda_4, t) >$ is similar to equation (49).

The entanglement of four spin-1 particles is in general a more complex problem. To characterize the entanglement of the state of four spin-1 particles, we can compute the entropy for the reduced density matrices of single site, blocks of two spin-1 sites and blocks of three spin-1 sites. In our case, the entropy of four spin-1 particle system is in general a complex function of the parameters $\lambda_3 t$ and $\lambda_4 t$. Figure 7 depicts the entropy for the reduced density matrices of single site. In Figure 8, we compute the entropy for the reduced density matrix of a block of two neighboring spin sites. Like the entropy for the reduced density matrix for single site, the variation of the plots of the entropy for neighboring sites with the parameters $\lambda_3 t$ and $\lambda_4 t$ attains minimum at the same parameter values. Moreover, we see that the entropies are not all independent. Indeed $S(\rho_{BC}) = S(\rho_B)$ and $S(\rho_{CD}) = S(\rho_D)$. We also note that $S(\rho_{AB})$ is not related in any way with the entropy of the reduced density matrix for single sites. However, if we consider blocks of three neighboring sites, we see that $S(\rho_{ABC}) = S(\rho_{AB})$. Also, $S(\rho_{BCD}) = S(\rho_A)$. The plots for the variations in the entropy for the reduced density matrices for three neighboring sites are shown in Figure 9. Thus, although the entropies for the reduced density matrices are not all independent, together they can give a rough good picture concerning the entanglement of the whole chain. In particular, we see from the relationships that the third site C does not seem to contribute significantly to the entanglement of the whole chain since $S(\rho_{CD}) = S(\rho_D)$ and $S(\rho_{ABC}) = S(\rho_{AB})$.

The concurrence for the four-particle system may be written as $2\sqrt{P(i,4,t)P(j,4,t)}(i < j = 1,2,3,4)$. These concurrence are not independent because of $P(2,4,t) = P(4,4,t)$. Moreover, the Schmidt decomposition of the four-particle system is more complicated because of the degenerate eigenvalues of the density matrices. This may be one of the reasons why the system of four-particle system will never reach a GHZ state.

As in the case of the three spin-1 particle states, the wave functions are also cyclic provided the following conditions are fulfilled by $\lambda_3 t = 2n\pi$ and $\lambda_4 t = 2m\pi; (n, m = 1, 2, ...)$, which give $\tan\frac{\phi}{4} = \frac{m}{n}$. A closed path is traced out by the parameter ϕ in the projective space under the cyclic evolution of the system. By using equation (49),

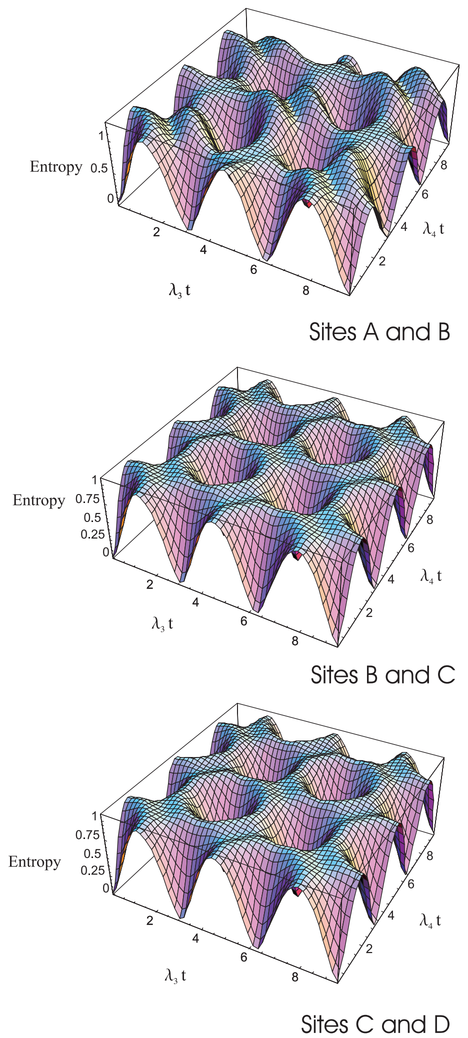

Fig. 8. The variation in the entropy of the reduced density matrix of a block of two neighboring sites in a four-spin-1 particle system.

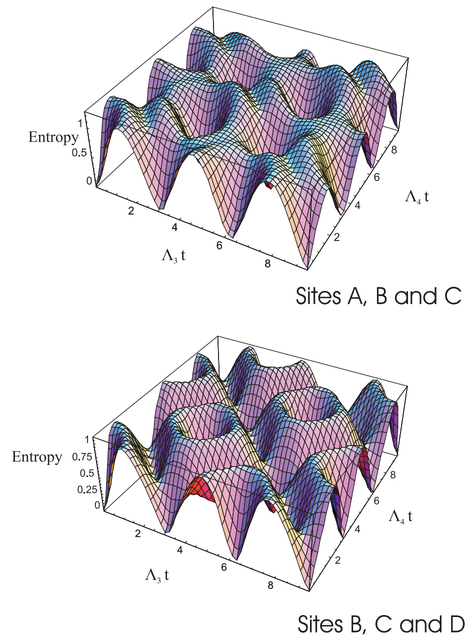

Fig. 9. The variation in the entropy of the reduced density matrix of a block of three neighboring sites in a four-spin-1 particle system.

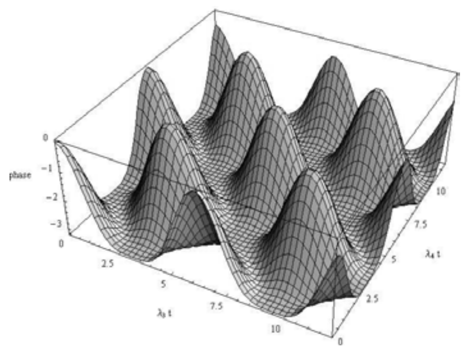

Fig. 10. Geometric phase for four spin-1 particle system. The geometric phase is changed as the state evolves from a pure state to a mixed state. The cyclic nature of geometric phase returns a memory of its motion.

we know that the system satisfies the condition of parallel transport. The Berry phase of the four spin-1 particle system is easily derived,

$$\gamma_g = i \oint_C \langle \psi(\lambda_3, \lambda_4, t) | d | \psi(\lambda_3, \lambda_4, t) \rangle = -\frac{\pi}{4} \left\{ 4 - [\cos(\lambda_4 t) + \cos(\lambda_3 t)]^2 \right\}. \quad (50)$$

The geometric phase for the system of four spin-1 particles as the function of $\lambda_3 t$ and $\lambda_4 t$ is exhibited in Figure 10. It is interesting to note that the entropy $S(\rho_A)$ and the geometric phase in the two and four spin-1 particle system have a relatively simpler form compared with the three spin-1 particle system.

We note that equation (26) between the entropy $S(\rho_A)$ and geometric phase is satisfied for the four particle system. Therefore, it appears that the spin-1 particles tend to form pairs under the spin-spin interaction. Moreover, the geometric phase is cyclically changed as the probability $P(1, 4, t)$ cyclically moves. The changed range of the geometric phase is from $-\pi$ to zero. At the minimum and maximum points $\gamma_g = -\pi, 0$, the corresponding entropy $S(\rho_A)$ is zero. When $\gamma_g = -\pi/2$, $S(\rho_A)$ is one. The cyclic nature of the geometric phase shows that the system may retain a memory of its motion.

5 N-particle system

Following our analysis in Sections 3 and 4, we can generalize our results to the N-particle system with $N-1$ particles in the longitudinal states and one particle in the transverse state. By using equations (14) and (15), one finds

$$
\begin{aligned}
|\psi_N(t)\rangle =& \frac{1}{N}\sum_{n=1}^{N}\sum_{k=1}^{N}\exp\left(i\frac{(2\pi k+\phi)(n-1)}{N}\right. \\
& \left. -i\lambda t\cos\frac{(2\pi k+\phi)}{N}\right) S_n^+|3\rangle^{\otimes N} \\
=& \sum_{n=1}^{N} e^{i\frac{N-1}{N}\phi} C_0\left(n,N,\lambda_N^c,\lambda_N^s,t\right) S_n^+|3\rangle^{\otimes N}, \quad (51)
\end{aligned}
$$

where the renormalized couplings, λ_N^c and λ_N^s, are

$$
\lambda_N^c = \lambda\cos\frac{\phi}{N}, \quad \lambda_N^s = \lambda\sin\frac{\phi}{N}, \tag{52}
$$

and

$$
\begin{aligned}
& C_0\left(n,N,\lambda_N^c,\lambda_N^s,t\right) = \frac{1}{N} \\
& \times\sum_{k=1}^{N}\exp\left(i\frac{(2\pi k)(n-1)}{N} - i\lambda_N^c t\cos\frac{2\pi k}{N} + i\lambda_N^s t\sin\frac{2\pi k}{N}\right).
\end{aligned} \tag{53}
$$

The probabilities of N-particle system may be written by

$$
P(n,N,t) = |C(n,N,t)|^2 = |C_0(n,N,t)|^2, \tag{54}
$$

where $C(n,N,t)$ is given by equation (15) and $P(1,N,t) + P(2,N,t) + ... + P(N,N,t) = 1$.

From Sections 3 and 4, we know that the entropy of the subsystem of a single site is important for two, three and four-particle system because of symmetry of our wave functions. Therefore, for simplification, we only consider the entropy of the subsystem of a single site for N-particle system. Extending equations (22) and (38), the entropy of the N-particle system is given by

$$
\begin{aligned}
S(\rho_A) = & -P(1,N,t)\log_2 P(1,N,t) - [1-P(1,N,t)] \\
& \times\log_2[1-P(1,N,t)].
\end{aligned} \tag{55}
$$

Because the wave function of N-particle system is a unit vector, the condition of parallel transport for the system can be similarly deduced [39]. Thus, the geometric phase of the N-particle system can be expressed by

$$
\begin{aligned}
\gamma_g(N) &= i\oint_C \langle\psi_N\left(\lambda_N^c,\lambda_N^s,t\right)|d|\psi_N\left(\lambda_N^c,\lambda_N^s,t\right)\rangle \\
&= -\frac{2\pi}{N^3}\sum_{n=1}^{N}(n-1)P\left(n,N,\lambda_N^c,\lambda_N^s,t\right).
\end{aligned} \tag{56}
$$

If we set $n = 2, 3, 4$, equation (56) will repeat the results of equations (25), (39) and (50), respectively. It is interesting to noted that the geometric phase is related to the probabilities of the N-particle system.

6 Noise and decoherence

The decay of quantum information due to the interaction of a system with its environment, which can be described by a superoperator. If the environment frequently scatters off the system, and the state of the environment is not monitored, then off-diagonal terms in the density matrix of the system decay rapidly in a preferred basis. The time scale for decoherence is set by the scattering rate, which may be much larger than the damping rate for the system. The great challenge facing any such information processing in the quantum regime lies in avoiding, controlling or overcoming the effects of decoherence.

When the relevant dynamical time scale of an open quantum system is long compared to the time for the environment to the forget quantum information, the evolution of the system is effectively local in time (the Markovian approximation). Much as general unitary evolution is generated by a Lindbladian $\mathcal{L}$ as described by the master equation [45],

$$
\frac{\partial\rho}{\partial t} = -i\left[\hat{\mathcal{H}},\rho\right] + \sum_{\mu}\left(L_\mu\rho L_\mu^+ - \frac{1}{2}\left\{L_\mu L_\mu^+,\rho\right\}\right), \tag{57}
$$

where the Lindblad operator, $L_\mu = \sqrt{\kappa_\mu(t)}S_\mu$, represent the coupling to the environment and S_μ is spin operators from equation (1). The decoherence time is approximately given by $1/\kappa_\mu(t)$. The noise can be controlled by switching on and off $\kappa_\mu(t)$. Now suppose the isotropic noise (L_x, L_y, and L_z) is applied to our system.

It is easy to find the matrix form of equation (57) in interaction picture as

$$
\begin{aligned}
& \frac{\partial}{\partial t}\begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{31} & \rho_{32} & \rho_{33} \end{pmatrix} = \\
& \kappa\begin{pmatrix} \rho_{22}+\rho_{33}-2\rho_{11} & -\rho_{21}-2\rho_{12} & -\rho_{31}-2\rho_{13} \\ -\rho_{12}-2\rho_{21} & \rho_{11}+\rho_{33}-2\rho_{22} & -\rho_{32}-2\rho_{23} \\ -\rho_{13}-2\rho_{31} & -\rho_{23}-2\rho_{32} & \rho_{11}+\rho_{22}-2\rho_{33} \end{pmatrix}.
\end{aligned} \tag{58}
$$

The analytic solution of equation (58) can be found. For diagonal elements,

$$
\begin{aligned}
\rho_{11}(t) =& \frac{1}{4}\left(2\rho_{11}(t_0)+\rho_{22}(t_0)+\rho_{33}(t_0)\right) \\
& + \frac{1}{4}\left(2\rho_{11}(t_0)-\rho_{22}(t_0)-\rho_{33}(t_0)\right)e^{-3\kappa(t-t_0)},
\end{aligned} \tag{59}
$$

$$
\begin{aligned}
\rho_{22}(t) =& \frac{1}{4}\left(2\rho_{22}(t_0)+\rho_{11}(t_0)+\rho_{333}(t_0)\right) \\
& + \frac{1}{4}\left(2\rho_{22}(t_0)-\rho_{11}(t_0)-\rho_{33}(t_0)\right)e^{-3\kappa(t-t_0)},
\end{aligned} \tag{60}
$$

$$
\begin{aligned}
\rho_{33}(t) =& \frac{1}{4}\left(2\rho_{33}(t_0)+\rho_{22}(t_0)+\rho_{11}(t_0)\right) \\
& + \frac{1}{4}\left(2\rho_{33}(t_0)-\rho_{22}(t_0)-\rho_{11}(t_0)\right)e^{-3\kappa(t-t_0)},
\end{aligned} \tag{61}
$$

for off-diagonal elements,

$$
\rho_{ij}(t) = \rho_{ij}(t_0)e^{-\kappa(t-t_0)}, i \neq j = 1,2,3. \tag{62}
$$

We see that the off-diagonal elements of the density matrix tend to decay in time as a simple exponential, with the decay constant denoting a relaxation time.

The properties of quantum information through noisy quantum channels are quantified by the fidelity which measures the overlap between the initial and time-developed state vectors. Now we take the density matrix in equation (21) as an initial state, The fidelity of the two-particle system is written as

$$\begin{aligned}F(t_0,t) =&\mathrm{Tr}\left(\rho_{in}(t_0)\rho_{out}(t)\right)\\ =&\frac{1}{4}\cos^2\lambda' t_0\left(1+\frac{1}{2}\cos^2\lambda' t_0\right.\\ &\left.-\left(1-\frac{3}{2}\cos^2\lambda' t_0\right)e^{-3\kappa(t-t_0)}\right)\\ &+\frac{1}{4}\sin^2\lambda' t_0\left(1+\sin^2\lambda' t_0\right.\\ &\left.+2\left(1-\frac{3}{2}\cos^2\lambda' t_0\right)e^{-3\kappa(t-t_0)}\right)\\ &+\frac{1}{2}\cos^4\lambda' t_0 e^{-\kappa(t-t_0)}.\end{aligned} \tag{63}$$

It is useful to calculate the average fidelity given by

$$\begin{aligned}F_{ad}(t_0,t) &= \frac{1}{4\pi}\int_0^{2\pi}d\phi\int_0^{\pi}F(t_0,t)\sin\lambda' t_0 d(\lambda' t_0)\\ &=\frac{49}{120}+\frac{28}{120}e^{-3\kappa(t-t_0)}+\frac{1}{10}e^{-\kappa(t-t_0)}.\end{aligned} \tag{64}$$

Similarly, the fidelities for the three-particle and four-particle systems can be obtained. Further, the fidelity for the N-particle system can be expressed as

$$\begin{aligned}F(N,t_0,t) =&\frac{1}{4}P(1,N,t_0)\left(1+\frac{1}{2}P(1,N,t_0)\right.\\ &\left.-\left(1-\frac{3}{2}P(1,N,t_0)e^{-3\kappa(t-t_0)}\right)\right.\\ &+\frac{1}{2}(1-P(1,N,t_0))\left(1-\frac{1}{2}P(1,N,t_0)\right.\\ &\left.+\left(1-\frac{3}{2}P(1,N,t_0)\right)e^{-3\kappa(t-t_0)}\right)\\ &+\frac{1}{2}P^2(1,N,t_0)e^{-\kappa(t-t_0)},\end{aligned} \tag{65}$$

which is related to the particle probabilities. For $N>2$ system, the average fidelity can be calculated by

$$F_{ad} = \frac{1}{2\pi^2}\int_0^{2\pi}d\phi\int_0^{\pi}d\theta_1\sin\theta_1\int_0^{\pi}F(N,t_0,t)\sin^2\theta_2 d\theta_2. \tag{66}$$

For $N=3$, $\theta_1=\lambda_1 t$ and $\theta_2=\lambda_2 t$, the average fidelity is

$$F_{ad} = 0.432+0.295e^{-3\kappa(t-t_0)}+0.028e^{-\kappa(t-t_0)}, \tag{67}$$

and for $N=4$, $\theta_3=\lambda_3 t$ and $\theta_4=\lambda_4 t$, the average fidelity is

$$F_{ad} = 0.446+0.339e^{-3\kappa(t-t_0)}+0.026e^{-\kappa(t-t_0)}. \tag{68}$$

It is noted that our fidelity is slightly different from one of spin-$\frac{1}{2}$, where two decay widths, $\sqrt{\kappa}$ and $\sqrt{3\kappa}$, are included.

7 Conclusion

An entangled composite system gives rise to nonlocal correlation between its subsystems that does not exist classically. This nonlocal property enables the local quantum operations and classical communication to transmit information with advantages no classical communication protocol can offer. The understanding of entanglement is thus at the very heart of quantum information theory. Many-particle system with strong quantum fluctuations is a challenging subject in quantum information. For the interacting system, the existence of entanglement is normal, the question is how to characterize the detail.

In present work, we have studied the spin-1 particle system under the spin-spin interaction between the transverse and the longitudinal polarization. It is known that the polarized vector boson production can be obtained by the collisions of the polarized hadron beams. Thus, by using a polarizing beam splitter similarly to references [46–48], we can separate the system of the transverse and the longitudinal polarization. In our studies, the eigenstates of the systems are obtained by generalized and extended Jordan-Wigner transformation, where expressions for the cyclic entanglement and cyclic geometric phase have been derived.

For the two-particle system, the wave function is described by a corresponding polar angle ($\theta=\lambda' t$) and azimuthal angle (ϕ). It is noted that the polar angle is variable of time. Thus the entanglement and geometric phase of the system is cyclically changed. Moreover, an exact equation is found between the entropy and geometric phase for two particles system.

System with N spin-1 particles ($N>2$), in contrast to a system of two spin-1 particles, evolves in time with two-parameters. Moreover, the entanglement and geometric phase related to the particle probabilities of a system of odd number of spin-1 particles is more complicating than a system with even number of spin-1 particles. In our model, the entropy of the subsystem of a single site can be taken as a measurement of entanglement for two and three-particle system.

In addition, the entropy of a single site, $S(\rho_A)$, and geometric phase for four particle system satisfy same equation (25) as the two particle system. We conjecture that the massive spin-1 particles under the spin-spin interaction are able to form pairs.

The entanglement and geometric phases of the N-particle are cyclically changed as the states evolve from a pure state to a mixed state. When the particle probabilities are equal each other, the system is in W-states. The W states have different from the GHZ states about properties of entanglement and geometric phase.

The work is done under the NUS Research Grant No. WBS: R-144-000-089-112 on Quantum Entanglement project.

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